Unit -

Uniform Convergence

B.Sc III

1. Cauchy's General Principle of Uniform Convergence:→

The Necessary and Sufficient Condition that the Sequence $\{f_n(x)\}$ should Converge uniformly on the set E is that Given any +ve number ε, there Exists an integer m, independent of x, such that for $n \geq m$ We have

$$|f_n(x) - f_{n+p}(x)| < \varepsilon \text{ for all } x \in E$$

Where p is any +ve integer

Proof :→ Necessary :→

Let the Sequence $\{f_n(x)\}$ Converges uniformly to the function $f(x)$ On E. then, for Given $\varepsilon > 0$, there Exists a +ve integer m such that

$$|f_n(x) - f(x)| < \varepsilon/2 \quad \text{for } n \geq m \text{ and } \forall x \in E$$

and hence $|f_{n+p}(x) - f(x)| < \varepsilon/2$ for $n \geq m$, a +ve integer p and $\forall x \in E$

Therefore,

$$|f_{n+p}(x) - f_n(x)| = |f_{n+p}(x) - f(x) + f(x) - f_n(x)|$$
$$\leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{for } n \geq m,\ x \in E$$

Hence the Condition is Necessary.

Sufficiency :→ Let $f_n(x)$ be any Sequence of function defined On E such that for Given $\varepsilon > 0$, there Exists a +ve integer m such that

$$|f_{n+p}(x) - f_n(x)| < \varepsilon \quad \text{———} \quad (1)$$

for all $n \geq m$ and for all $x \in E$, where p is a +ve integer.

Hence by Cauchy's general principle of convergence $\{f_n(x)\}$ tends to a limit, say $f(x)$, for each value of x in E,

i.e. $\lim_{p \to \infty} f_p(x) = f(x)$

Thus keeping n fixed and letting $p \to \infty$ in (1), we have

$$|f_n(x) - f(x)| < \epsilon$$

when $n \geq m$, for all $x \in E$

Hence the convergence is uniform.

Corollary :- A series $\sum u_n(x)$ will converge uniformly on E if and only if for given $\epsilon > 0$, there exists a +ve integer m such that

$$|u_{n+1}(x) + u_{n+2}(x) + \dots + u_{n+p}(x)| < \epsilon$$

for all $n \geq m$ and for all $x \in E$ where p is a +ve integer.

TEST for Uniform and Absolute Convergence :-

1. Dini's Theorem :-

Given a series $\sum u_n(x)$ such that

(i) each term is non-negative and continuous over an interval $[a,b]$

(ii) The series converges to $f(x)$ over $[a,b]$

(iii) $f(x)$ is continuous over $[a,b]$

Proof :- Let $\epsilon > 0$ be given, we choose any point $t \in [a,b]$ and as $f(x)$ is continuous over $[a,b]$

therefore $f(x)$ is continuous at t. Now there exists a nhd. U_1 of t such that

$$|f(x) - f(t)| < \epsilon/3 \text{ for } x \in [a,b] \cap U_1 \quad \text{——— (i)}$$

Again since $\Sigma u_n(x)$ is convergent to $f(x)$ for each $x \in [a,b]$ there exists a +ve integer n_1 such that

$$|f(t) - f_n(t)| < \epsilon/3 \text{ for all } n \geq n_1.$$

Where $f_n(x)$ is the partial sum of $u_n(x)$

i.e. $f_n(x) = \sum_{r=1}^{n} u_r(x)$

Again as $u_1, u_2, \ldots u_n$ are all real all continuous over $[a,b]$ there its partial sum $f_n(x)$ is also continuous over $[a,b]$. Hence there exists a nhd U_2 of t such that

$$|f_n(t) - f_n(x)| < \epsilon/3 \text{ for } x \in [a,b] \cap U_2 \quad \text{—— (ii)}$$

Let $U = U_1 \cap U_2$, then

$$|f(x) - f_n(x)| = |f(x) - f(t) + f(t) - f_n(t) + f_n(t) - f_n(x)|$$

$$\leq |f(x) - f(t)| + |f(t) - f_n(t)| + |f_n(t) - f_n(x)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad \text{——— (iii)}$$

for all x in U and for all $n \geq n_1$

Now as $t \in [a,b]$, we can obtain a collection $\{U(t)\}$ of nhd. which covers $[a,b]$ and for which relation (iii) holds.

Hence by Heine-Borel theorem, we can choose a finite no of nhds $U(t_1), U(t_2), \ldots U(t_k)$ such that these n'hoods cover $[a,b]$.

Let the corresponding integers be $n_1, n_2, n_3, \ldots n_k$ respectively and let m be the maximum of.

Since . . .

Hence (ii) holds,

$|f(x) - f_n(x)| < \epsilon$ for $n \geq m$ and for all x in $(S_1), (S_2), \ldots (S_n)$ and uniformly for all x over $[a,b]$

Hence the series $\sum u_n(x)$ converges uniformly to $f(x)$.

M_n-TEST → ~~Let $\{f_n(x)\}$ converges~~ uniformly.

~~Let~~

Let $\{f_n\}$ be a sequence of real valued functions defined on a set E

Let $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in E$

and let $M_n = \sup\{|f_n(x) - f(x)| : x \in E\}$

then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$

Proof:→ Necessary:→ Let $\{f_n\}$ converges uniformly to f on E. then for given $\epsilon > 0$, there exists a +ve integer m such that

~~Hence~~ ~~for $n \geq m$~~ $|f_n(x) - f(x)| < \epsilon$

i.e. for $n \geq m$ and $\forall x \in E$

$\therefore \sup|f_n(x) - f(x)| \leq \epsilon$ for $n \geq m$ and for all $x \in E$

Hence $\{f_n(x)\}$ converges uniformly to f on E.

Imp Weierstrass's M-TEST :→

Statement :→

The Series $\Sigma U_n(x)$ is uniformly as well as absolutely Convergent on the Set E, if for all values of x in E $|U_n(x)| < M_n$

where M_n is independent of x

And ΣM_n is Convergent

Proof :→ As $|U_n(x)| < M_n$ and ΣM_n is Convergent

hence By Comparison Test $\Sigma |U_n(x)|$ is Convergent

i.e $U_n(x)$ is absolutely Convergent

Again $|U_{n+1}(x) + U_{n+2}(x) + \dots + U_{n+p}(x)| \leq |U_{n+1}(x)| + |U_{n+2}(x)| + \dots + |U_{n+p}(x)| < M_{n+1} + M_{n+2} + \dots + M_{n+p}$

But since ΣM_n is Convergent

Given $\epsilon > 0$ there exists a +ve integer m (Independent of x as M_n is independent of x) Such that

$M_{n+1} + M_{n+2} + \dots + M_{n+p} < \epsilon, \; n > m$

$\therefore \; |U_{n+1}(x) + U_{n+2}(x) + \dots + U_{n+p}(x)| < \epsilon$ for $n > m$

where m is independent of x

Hence by Necessary and Sufficient Condition $\Sigma U_n(x)$ is uniformly Convergent

Q. State and Prove

Abel's Test for Uniform Convergence.

Statement → The series $\sum u_n(x) v_n(x)$ Converges uniformly for all $x \in [a,b]$ if

(i) $\{v_n(x)\}$ is a +ve ~~dec~~ decreasing Sequence for all values of $x \in [a,b]$

(ii) $\sum u_n$ is uniformly Convergent

(iii) $v_1(x)$ is bounded for all $x \in [a,b]$

Proof → Let $\epsilon > 0$ be given.

Let the partial sum $\sum_{r=1}^{n} u_r(x)$ be ~~bounded~~ denoted by $S_n(x)$.

As $\sum u_n(x)$ is uniformly Convergent, we can find an integer, So that whatever positive integer p may be

u_{m+1}, $u_{m+1} + u_{m+2}$ - - - -, $u_{m+1} + u_{m+2} + \cdots + u_{m+p}$ are Numerically less than ϵ. thus by using Abel's Lemma.

we Find that

$$\left| \sum_{n=m+1}^{m+p} u_n(x) v_n(x) \right| \le |u_{m+1}(x)|\,|v_{m+1}(x)| + \cdots + |u_{m+p}|\,|v_{m+p}|$$

$$\le \left[|u_{m+1}(x)| + |u_{m+2}(x)| + \cdots + |u_{m+p}(x)| \right] v_m(x)$$

$$< \epsilon\, v_m(x) < \epsilon K \quad \text{——— (1)}$$

Where $v_m(x) \le v_1(x) < K$ (Say)

[Since $v_1(x)$ is bounded for all $x \in [a,b]$ and v_n is a +ve decreasing Sequence].

thus Necessary Condition, (1) gives that $\sum u_n(x) v_n(x)$ Converges Uniformly in $[a,b]$